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Formal topological characterizations of various continuous domains[☆]

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Abstract

In this paper, some new concepts such as (super-compact) quasi-bases and the consistently (locally) coherent property of super-compact quasi-bases, are introduced. With these concepts, various continuous domains including bc-domains and (s)L-domains, are successfully characterized in formal topological ways. Furthermore, to deal with algebraic domains, the concept of quasi-formal points, a generalization of formal points, is introduced. Formal topological characterizations of various algebraic domains via quasi-formal points are obtained.

Keywords: Continuous domain; Locally super-coherent topology; Quasi-base; (Quasi-)formal point; BC-domain; L-domain

1. Introduction

In 1972, Dana Scott introduced a class of lattices called continuous lattices in order to provide models for the type free calculus in logic (see [1]). Later, a more general notion of continuous domains, which are mathematical structures used in semantics as carriers of meaning, was introduced in [2]. Now domain theory has received more and more attention (see [3–8]) of both mathematicians and computer scientists. And the study of domain theory integrates the studies of order structures, topological structures as well as algebraic structures. Meanwhile, domain theory is also closely related to theoretical computer science and yields many applications in various areas. It should be noted that a distinctive feature of the theory of continuous domains is that many of the considerations are closely interlinked with topological ideas (see [3,4,8,9]).

The Scott topology, as an order-theoretical topology, is of fundamental importance in domain theory. It lies at the heart of the structure of continuous domains. In [2], Lawson proved that a dcpo is continuous if and only if its Scott topology is completely distributive. However, the definitions or characterizations of the Scott topology always assume the underlying set to be equipped with a partial order.

Domain theory can be seen as a branch of formal topology which is the topology as developed in (Martin Löf's) type theory. Recently, Sambin in [10], introduced the concept of super-coherent topology. A purely topological characterization of the Scott topologies over algebraic dcpos, independently of any orders, was presented.

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Since continuous domains are more general and complicated than algebraic ones, it is tempting to find similar topological characterizations of various kinds of continuous domains. For this purpose, in this paper, a series of new concepts such as the locally super-coherent topology, (super-compact) quasi-bases and the consistently (resp., locally) coherent property of super-compact quasi-bases, are introduced. With these concepts, various continuous domains including bc-domains, sL-domains and L-domains, are successfully characterized in formal topological ways. Furthermore, to deal with algebraic domains, the concept of quasi-formal points, a generalization of formal points in some sense, is introduced. And formal topological characterizations of various algebraic domains via quasi-formal points are also thus obtained.

2. Preliminaries

Recall that in a poset P, a subset $D \subseteq P$ is *directed* if each finite subset of D has an upper bound in D. Any directed lower set of P is called an *ideal*. The set of all ideals of P ordered by set inclusion is called the *ideal completion* of P, denoted by $Idl(P)$. A poset is called a *directed complete poset* (briefly *dcpo*) if each directed subset has a supremum. A poset is said to be *bounded complete* if every subset with an upper bound has a supremum. This implies that a bounded complete poset has a bottom. A *complete lattice* is a poset in which every subset has a supremum.

Let P be a dcpo and $x, y \in P$. We say that x *approximates* y, written $x \ll y$, if whenever D is directed with $\sup D \geq y$, then $x \leq d$ for some $d \in D$. An element $k \in P$ is *compact* if $k \ll k$. The set of all compact elements of P is denoted by $K(P)$. If for every element $x \in P$, the set $\Downarrow x := \{a \in P : a \ll x\}$ is directed and $\sup \Downarrow x = x$, then P is called a *continuous domain*. A dcpo is called an *algebraic domain* if every element is the directed supremum of compact elements that approximate it. A bounded complete continuous (algebraic) domain is called a *bc-domain* (*Scott domain*). A(n) *continuous* (*algebraic*) *lattice* means a(n) continuous (algebraic) domain which is a complete lattice. An (algebraic) *sL-domain* (see [6]) is a(n) continuous (algebraic) domain in which every principal ideal is a join-semilattice. An (algebraic) *L-domain* is a(n) continuous (algebraic) domain in which every principal ideal is a complete lattice.

An upper set U of a dcpo P is said to be *Scott open* if for all directed sets $D \subseteq P$, $\sup D \in U$ implies $U \cap D \neq \emptyset$. The set of Scott open sets of P forms a topology, called the *Scott topology* and denoted by $\sigma(P)$.

Proposition 2.1 (*See [4]*). *If P is a continuous domain, then the approximating relation $\ll$ has the interpolation property (INT): $x \ll z \Rightarrow \exists y \in P$ such that $x \ll y \ll z$.*

Proposition 2.2 (*See [4]*). *Let P be a continuous domain. Then for each $x \in P$, the set $\Uparrow x = \{y \in P : x \ll y\}$ is Scott open, and these form a base for the Scott topology.*

A subset F of a space X is said to be *irreducible*, if $F \neq \emptyset$ and for any pair of closed sets F_1 and F_2 in X, $F \subseteq F_1 \cup F_2$ implies that $F \subseteq F_1$ or $F \subseteq F_2$. A T_0-space is said to be *sober* if every irreducible closed set is a closure of a unique point. It is known that continuous domains equipped with Scott topologies are sober (see [4, Corollary II-1.12]).

A topology $\mathcal{O}(X)$ induces a preorder, called the *specialization order* which is defined by $\forall u, v \in X$, $u \leq_X v \Leftrightarrow u \in cl_X(\{v\}) \Leftrightarrow \forall U \in \mathcal{O}(X), u \in U \rightarrow v \in U$, where $cl_X(\{v\})$ is the closure of the set $\{v\}$ in the space $(X, \mathcal{O}(X))$. The ordered set $(X, \leq_X)$ is denoted by ΩX in short. Note that every open set of X is an upper set of ΩX and the closure of $\{v\}$ is the lower set $\downarrow v = \{x \in X : x \leq_X v\}$. It is clear that $(X, \mathcal{O}(X))$ is a T_0-space iff $\leq_X$ is a partial order.

A T_0-space X is called a *monotone convergence space* (see [4,11]) if every subset D directed relative to the specialization order has a supremum, and the relation $\sup D \in U$ for any open set U of X implies $D \cap U \neq \emptyset$.

Remark 2.3 (*See [4]*). Each sober space is a monotone convergence space and every monotone convergence space X is a dcpo in its specialization order with $\mathcal{O}(X) \subseteq \sigma(\Omega X)$.

Lemma 2.4 (*See [4]*). *For a monotone convergence space $(X, \mathcal{O}(X))$ and its order of specialization, the following conditions are equivalent:*

(1) *ΩX is a continuous domain and the topology of X is the Scott topology;*
(2) *$\mathcal{O}(X)$ is a completely distributive lattice.*

Definition 2.5 (*see [12]*). Let L be a complete lattice. (1) The *complete way-below* relation $\lhd$ in L is defined for all $a, b \in L$, $a \lhd b \Leftrightarrow (\forall S \subseteq L,\ b \leq \sup S \Rightarrow \exists s \in S, a \leq s)$; (2) If $a \in L$ and $B \subseteq \{t \in L : t \lhd a\}$ with $\sup B = a$, then B is called an *approximating complete way-below set* of a in L. We will use $B(a)$ to denote any one of the approximating complete way-below sets of $a \in L$.

Lemma 2.6 (*See [12]*). *Let L be a complete lattice. Then L is completely distributive if and only if $\forall a \in L$, a has an approximating complete way-below set.*

3. Locally super-coherent topologies and characterizations of continuous domains

In this section, we introduce the new concept of locally super-coherent topology and present a purely topological characterization of continuous domains.

The following definition comes from [10].

Definition 3.1 (*See [10]*). Let $(X, \mathcal{O}(X))$ be a topological space. Then a subset U is said to be *super-compact* if for any family of open sets $\{V_i\}_{i\in I}$ with $U \subseteq \bigcup_{i\in I} V_i$, there is $i_0 \in I$ such that $U \subseteq V_{i_0}$. The topology $\mathcal{O}(X)$ is called *super-coherent* if it is sober and has a base of super-compact open sets.

Theorem 1 of [10] says that any super-coherent space $(X, \mathcal{O}(X))$ coincides with the Scott topology of a suitable algebraic domains over X. Since algebraic domains with Scott topologies are all coherent, Theorem 1 of [10], as well as Theorem 6.4 below in this paper, justifies the choice of the terminology of super-coherent.

Definition 3.2. A topological space $(X, \mathcal{O}(X))$ is called *locally super-compact* if every point of X has a base of (not necessarily open) super-compact neighborhoods. The topology $\mathcal{O}(X)$ is called *locally super-coherent* if it is sober and locally super-compact. Clearly, every super-coherent topology is locally super-coherent.

Proposition 3.3. *Let P be a continuous domain. Then $(P, \sigma(P))$ is a locally super-coherent space.*

Proof. It is known that $(P, \sigma(P))$ is sober (see [4, Corollary II-1.12]). Let U be a Scott open set containing x. Since P is a continuous domain, there is $y \ll x$ such that $y \in U$ and $x \in \Uparrow y \subseteq \uparrow y \subseteq U$. This shows that $\uparrow y$ is a super-compact neighborhood of x and thus $\{\uparrow t : t \ll x\}$ is a base of super-compact neighborhoods of x. So $(P, \sigma(P))$ is a locally super-coherent space. $\square$

Proposition 3.4. *Let $(X, \mathcal{O}(X))$ be a locally super-compact space. Then the topology $\mathcal{O}(X)$ is a completely distributive lattice.*

Proof. By Lemma 2.6, it suffices to prove that every element U of $\mathcal{O}(X)$ has an approximating complete way-below set. If $U = \emptyset \in \mathcal{O}(X)$, then $B(\emptyset) = \emptyset$ is an approximating complete way-below set of U. Next we consider the case of non-empty U. For each $x \in U$, since X is locally super-compact, there is a super-compact neighborhood N_x such that $x \in \text{int } N_x \subseteq N_x \subseteq U$ and $U = \bigcup_{x\in U} \text{int } N_x$, where $\text{int } N_x$ denotes the interior of the set N_x. Let $B(U) = \{\text{int } N_x : x \in U\}$. Then (1) $U = \bigcup B(U)$; (2) If $\{V_\alpha\}_{\alpha\in\Gamma}$ is another family of open sets with $\bigcup_{\alpha\in\Gamma} V_\alpha \supseteq U$, by the super-compactness of N_x, there is $\alpha_x \in \Gamma$ such that $\text{int } N_x \subseteq N_x \subseteq V_{\alpha_x}$ for all $x \in U$. This shows that $\text{int } N_x \lhd U$ for all $x \in U$ and $B(U)$ is an approximating complete way-below set of U. By Lemma 2.6, $\mathcal{O}(X)$ is a completely distributive lattice. $\square$

Theorem 3.5. *For any T_0 space $(X, \mathcal{O}(X))$ and its order of specialization, the following conditions are equivalent:*

(1) *X is a locally super-coherent space;*
(2) *X is a locally super-compact, monotone convergence space;*
(3) *ΩX is a continuous domain and the topology of X is the Scott topology.*

Proof. (1) $\Rightarrow$ (2): Apply Remark 2.3.

(2) $\Rightarrow$ (3): Apply Proposition 3.4 and Lemma 2.4.

(3) $\Rightarrow$ (1): This follows from Proposition 3.3. $\square$

The equivalence of (1) and (3) in Theorem 3.5 shows that a topology is locally super-coherent if and only if it is the Scott topology over a suitable continuous domain. So, a purely topological characterization of Scott topologies over continuous domains is obtained, generalizing the result of [10, Theorem 1] for algebraic domains.

Corollary 3.6. *A dcpo P is continuous iff $(P, \sigma(P))$ is locally super-compact.*

Proof. $\Rightarrow$: Apply Proposition 3.3.
$\Leftarrow$: This follows from Theorem 3.5 and the fact that any dcpo with the Scott topology is a monotone convergence space. $\square$

4. Formal points and (super-compact) quasi-bases

Given a topological space X with a base $\mathcal{B}$, the set $Pt(\mathcal{B})$ of *formal points* (see [10]) of the topology $\mathcal{O}(X)$ is defined to be the set of all proper filters α of $\mathcal{B}$ (in the set inclusion order) satisfying for all $V \in \alpha$ and $V_i \in \mathcal{B}(i \in I)$ with $V \subseteq \bigcup_{i \in I} V_i$, there is $i_0 \in I$ such that $V_{i_0} \in \alpha$.

The *canonical* map $\phi : X \to Pt(\mathcal{B})$ is defined by $\phi(x) = \{U \in \mathcal{B} : x \in U\}$ for all $x \in X$. It is straightforward to show that, for all $x \in X$, the set $\phi(x) = \{U \in \mathcal{B} : x \in U\}$ is a formal point. Clearly, ϕ is injective iff $\mathcal{O}(X)$ is T_0.

Lemma 4.1. *Let $(X, \mathcal{O}(X))$ be a space with bases $\mathcal{B}$ and $\mathcal{B}_1$. Then*

(1) $(Pt(\mathcal{O}(X)), \subseteq) \cong (Pt(\mathcal{B}), \subseteq) \cong (Pt(\mathcal{B}_1), \subseteq)$*;*
(2) *If $(X, \mathcal{O}(X))$ is sober, then the canonical map $\phi : X \to Pt(\mathcal{B})$ is a bijection.*

Proof. (1) Define $u : Pt(\mathcal{O}(X)) \to Pt(\mathcal{B})$ such that $u(\alpha^*) = \alpha^* \cap \mathcal{B}$ for all $\alpha^* \in Pt(\mathcal{O}(X))$. Define $v : Pt(\mathcal{B}) \to Pt(\mathcal{O}(X))$ such that for all $\alpha \in Pt(\mathcal{B})$, $v(\alpha) = \{U \in \mathcal{O}(X) : \exists B \in \alpha \text{ such that } B \subseteq U\}$. It is easy to show that u and v are meaningful, order preserving and mutually inverse to each other.

(2) By (1), it suffices to show that $\phi : X \to Pt(\mathcal{O}(X))$ is a bijection. Since soberity implies T_0, ϕ is injective. To show that ϕ is surjective, for any $\alpha^* \in Pt(\mathcal{O}(X))$, let $W = \bigcup(\mathcal{O}(X) - \alpha^*) \in \mathcal{O}(X)$. It follows from $\alpha^* \in Pt(\mathcal{O}(X))$ that $W \notin \alpha^*$. We claim that $X - W$ is irreducible closed. To show this, let F_1, F_2 be non-empty closed sets with $X - W \subseteq F_1 \cup F_2$. Then $(X - F_1) \cap (X - F_2) \subseteq W \notin \alpha^*$. Thus, we have $X - F_1 \notin \alpha^*$ or $X - F_2 \notin \alpha^*$. This means that $X - F_1 \subseteq W$ or $X - F_2 \subseteq W$, i.e., $X - W \subseteq F_1$ or $X - W \subseteq F_2$. So, $X - W$ is irreducible closed. By soberity, there is a unique $x \in X$ such that $X - W = cl_X(\{x\})$. For this x, it is easy to see that $x \notin W$ and $\phi(x) = \alpha^*$. So, ϕ is surjective, as desired. $\square$

Theorem 4.2. *For a sober space X with a base $\mathcal{B}$, the canonical map $\phi : (X, \leq_X) \to (Pt(\mathcal{B}), \subseteq)$ is an order isomorphism.*

Proof. By Lemma 4.1(2), ϕ is bijective. That ϕ is monotone follows from the fact that every open set is an upper set in the specialization order. To show that ϕ^{-1} is also monotone, suppose that $x \not\leq_X y$. Then $x \in X \setminus cl_X(\{y\}) \in \mathcal{O}(X)$ and there is $U \in \phi(x)$ such that $x \in U \subseteq X \setminus cl_X(\{y\})$. Noticing that $y \notin X \setminus cl_X(\{y\})$, we have $U \notin \phi(y)$ and $\phi(x) \not\subseteq \phi(y)$. So, ϕ^{-1} is monotone and ϕ is an order isomorphism. $\square$

By Theorem 4.2, we immediately have

Definition 4.3. A *quasi-base* for a topological space $(X, \mathcal{O}(X))$ is a family $q\mathcal{B}$ of subsets of X satisfying the following conditions:

(1) For all $U \in q\mathcal{B}$, $U \neq \emptyset$;
(2) The family $q\mathcal{B}^\circ = \{\text{int } U : U \in q\mathcal{B}\}$ is a base of the space X, i.e., for all $x \in V \in \mathcal{O}(X)$, there is $U \in q\mathcal{B}$ such that $x \in \text{int } U \subseteq U \subseteq V$.

A topological space X is said to have a *super-compact quasi-base* if X has a quasi-base consisting of super-compact sets.

Remark 4.4. Given a topological space $(X, \mathcal{O}(X))$ with a base $\mathcal{B}$ consisting of super-compact open sets, then the family $\mathcal{B}^* = \{U \in \mathcal{B} : U \neq \emptyset\}$ is both a base consisting of non-empty super-compact open sets and a super-compact quasi-base of X.

Proposition 4.5. *A topological space X has a super-compact quasi-base iff X is locally super-compact.*

Proof. Straightforward. $\square$

Theorem 4.6. *Let X be a sober space with a super-compact quasi-base $q\mathcal{B}$. Then $(Pt(q\mathcal{B}^{\circ}), \subseteq)$ is a continuous domain.*

Proof. The theorem follows from Proposition 4.5, Theorems 3.5 and 4.2. $\square$

Proposition 4.7. *Let X be a topological space with a super-compact quasi-base $q\mathcal{B}$ and $U \in q\mathcal{B}$. Then the set $\Uparrow U := \{\mathrm{int}\, V : U \subseteq \mathrm{int}\, V \subseteq V \in q\mathcal{B}\}$ is a formal point with respect to the base $q\mathcal{B}^{\circ} = \{\mathrm{int}\, W : W \in q\mathcal{B}\}$.*

Proof. We divide the proof into the following three steps:

(1) It is clear that $\emptyset \notin \Uparrow U$. Since $q\mathcal{B}^{\circ}$ is a base of the space X, there is a family $\{\mathrm{int}\, B_i : B_i \in q\mathcal{B}$ and $i \in I\} \subseteq q\mathcal{B}^{\circ}$ such that $X = \bigcup_{i \in I} \mathrm{int}\, B_i$. Then we have $U \subseteq X = \bigcup_{i \in I} \mathrm{int}\, B_i$. By the super-compactness of U, there is $i_0 \in I$ such that $U \subseteq \mathrm{int}\, B_{i_0} \subseteq B_{i_0} \in q\mathcal{B}$. This shows that $\mathrm{int}\, B_{i_0} \in \Uparrow U$ and thus $\Uparrow U \neq \emptyset$.

(2) Let $\mathrm{int}\, V, \mathrm{int}\, W \in \Uparrow U$ with $V, W \in q\mathcal{B}$. Then we have $U \subseteq \mathrm{int}\, V \cap \mathrm{int}\, W$. Since $q\mathcal{B}^{\circ}$ is a base, there is a family $\{\mathrm{int}\, B_j : B_j \in q\mathcal{B}$ and $j \in J\} \subseteq q\mathcal{B}^{\circ}$ such that $\mathrm{int}\, V \cap \mathrm{int}\, W = \bigcup_{j \in J} \mathrm{int}\, B_j$. By the super-compactness of U, there is $j_0 \in J$ such that $U \subseteq \mathrm{int}\, B_{j_0} \subseteq B_{j_0} \in q\mathcal{B}$. This shows that there is $\mathrm{int}\, B_{j_0} \in \Uparrow U$ such that $\mathrm{int}\, B_{j_0} \subseteq \mathrm{int}\, V \cap \mathrm{int}\, W$.

(3) Let $\mathrm{int}\, V \in \Uparrow U$ and $V_i \in q\mathcal{B} (i \in I)$ with $\mathrm{int}\, V \subseteq \bigcup_{i \in I} \mathrm{int}\, V_i$. Then we have $U \subseteq \mathrm{int}\, V \subseteq \bigcup_{i \in I} \mathrm{int}\, V_i$. By the super-compactness of U, there is $i_0 \in I$ such that $U \subseteq \mathrm{int}\, V_{i_0} \subseteq V_{i_0} \in q\mathcal{B}$. This shows that $\mathrm{int}\, V_{i_0} \in \Uparrow U$.

By steps (1)–(3), $\Uparrow U$ is a formal point with respect to the base $q\mathcal{B}^{\circ}$. $\square$

The following proposition gives more detailed property for topological spaces with super-compact quasi-bases.With this proposition, one can obtain Theorem 4.6 in another way. And the proposition will be useful in what follows.

Proposition 4.8. *Let X be a topological space with a super-compact quasi-base $q\mathcal{B}$ and $x \in X$. Then $\Uparrow U \ll \phi(x)$ for all* $\mathrm{int}\, U \in \phi(x) = \{\mathrm{int}\, U : x \in \mathrm{int}\, U \in q\mathcal{B}^{\circ}\}$ *and* $\phi(x) = \bigcup_{\mathrm{int}\, U \in \phi(x)} \Uparrow U$.

Proof. Let $\{\alpha_i : i \in I\}$ be a directed family of formal points w.r.t the base $q\mathcal{B}^{\circ}$ with $\sup_{i \in I} \alpha_i \supseteq \phi(x)$. Then it is straightforward to verify that $\bigcup_{i \in I} \alpha_i$ is a formal point. So, $\phi(x) \subseteq \sup_{i \in I} \alpha_i = \bigcup_{i \in I} \alpha_i$. Then for all $\mathrm{int}\, U \in \phi(x)$, there is $i_0 \in I$ such that $\mathrm{int}\, U \in \alpha_{i_0}$. Let $\mathrm{int}\, V \in \Uparrow U$. We have $\mathrm{int}\, V \supseteq U \supseteq \mathrm{int}\, U \in \alpha_{i_0}$ and thus $\mathrm{int}\, V \in \alpha_{i_0}$. This shows that $\Uparrow U \subseteq \alpha_{i_0}$ and $\Uparrow U \ll \phi(x)$. It is clear that $\phi(x) \supseteq \bigcup_{\mathrm{int}\, U \in \phi(x)} \Uparrow U$. Suppose $\mathrm{int}\, W \in \phi(x)$. Then by Definition 4.3(3), there is $U_0 \in q\mathcal{B}$ such that $x \in \mathrm{int}\, U_0 \subseteq U_0 \subseteq \mathrm{int}\, W$. This shows that $\mathrm{int}\, U_0 \in \phi(x)$ and $\mathrm{int}\, W \in \Uparrow U_0 \subseteq \bigcup_{\mathrm{int}\, U \in \phi(x)} \Uparrow U$. So, we have $\phi(x) = \bigcup_{\mathrm{int}\, U \in \phi(x)} \Uparrow U$, as desired. $\square$

5. Characterizations of BC-domains and L-domains

In this section, we present purely topological characterizations of bc-domains and L-domains by the technique of formal points and (super-compact) quasi-bases.

Definition 5.1. A super-compact quasi-base $q\mathcal{B}$ for a topological space X is said to have the *consistently coherent property* if for all $U, V \in q\mathcal{B}$,

(1) $U \cap V \neq \emptyset$ implies that $U \cap V \in q\mathcal{B}$;
(2) If $\emptyset \neq U \cap V \subseteq \mathrm{int}\, W \subseteq W \in q\mathcal{B}$, then there are $U_1, V_1 \in q\mathcal{B}$ such that $U \subseteq \mathrm{int}\, U_1$, $V \subseteq \mathrm{int}\, V_1$ and $U \cap V \subseteq \mathrm{int}\, U_1 \cap \mathrm{int}\, V_1 \subseteq U_1 \cap V_1 \subseteq \mathrm{int}\, W$.

The following theorem is one of main theorems which characterizes bc-domains in a purely topological way, generalizing the result of [10, Theorem 3] for Scott domains.

Theorem 5.2. *Let $(X, \mathcal{O}(X))$ be a sober space with a super-compact quasi-base $q\mathcal{B}$ and $X \in q\mathcal{B}$. If $q\mathcal{B}$ has the consistently coherent property, then $(Pt(q\mathcal{B}^{\circ}), \subseteq)$ is a bc-domain. Conversely, any bc-domain can be obtained in this way.*

Proof. Clearly, the set $\{X\}$ is a formal point and thus $(Pt(q\mathcal{B}^\circ), \subseteq)$ has a bottom. By Theorem 4.6, $(Pt(q\mathcal{B}^\circ), \subseteq)$ is a continuous domain. Let $\alpha, \beta \in Pt(q\mathcal{B}^\circ)$ with an upper bound γ. Since X is sober, there are $x, y, z \in X$ such that $\phi(x) = \alpha$, $\phi(y) = \beta$ and $\phi(z) = \gamma$, where the map ϕ is defined as in Proposition 4.8. By Proposition 4.8, $\alpha = \bigcup_{\text{int } U \in \alpha} \Uparrow U$ and $\beta = \bigcup_{\text{int } V \in \beta} \Uparrow V$ are both directed unions. To show that α and β has a supremum in $(Pt(q\mathcal{B}^\circ), \subseteq)$, it suffices to show that for all int $U \in \alpha$ and int $V \in \beta$, formal points $\Uparrow U$ and $\Uparrow V$ has a supremum. Since γ is an upper bound of α and β, we have int U, int $V \in \gamma = \phi(z)$. This shows that $z \in \text{int } U \cap \text{int } V \subseteq U \cap V$. Since the super-compact quasi-base $q\mathcal{B}$ has the consistently coherent property, we have $U \cap V \in q\mathcal{B}$. By Proposition 4.7, $\Uparrow (U \cap V) \in Pt(q\mathcal{B}^\circ)$. Clearly, $\Uparrow (U \cap V)$ is an upper bound of formal points $\Uparrow U$ and $\Uparrow V$. Let $\xi \in Pt(q\mathcal{B}^\circ)$ be any upper bound of $\Uparrow U$ and $\Uparrow V$. For all int $W \in \Uparrow (U \cap V)$, we have $U \cap V \subseteq \text{int } W$. By Definition 5.1(2), there are $U_1, V_1 \in q\mathcal{B}$ such that $U \subseteq \text{int } U_1$, $V \subseteq \text{int } V_1$ and $U \cap V \subseteq \text{int } U_1 \cap \text{int } V_1 \subseteq U_1 \cap V_1 \subseteq \text{int } W$. Since int $U_1 \in \Uparrow U$, int $V_1 \in \Uparrow V$ and ξ is a filter of $(q\mathcal{B}^\circ, \subseteq)$, we have int $W \in \xi$ and $\Uparrow (U \cap V) \subseteq \xi$. This shows that $\Uparrow (U \cap V)$ is the supremum of $\Uparrow U$ and $\Uparrow V$ and thus α and β has a supremum. So, $(Pt(q\mathcal{B}^\circ), \subseteq)$ is a bc-domain.

Conversely, let P be a bc-domain. By Proposition 2.2, it is straightforward to verify that $q\mathcal{B}' = \{\uparrow x : x \in P\}$ is a super-compact quasi-base for $\sigma(P)$. Since P has a bottom $\perp$, we have $P = \uparrow \perp \in q\mathcal{B}'$. Next we show that $q\mathcal{B}'$ has the consistently coherent property. Let $x, y \in P$.

(1) If $\uparrow x \cap \uparrow y \neq \emptyset$, then by the bounded completeness of P, the supremum of x and y exists, denoted by $x \vee y$. So we have $\uparrow x \cap \uparrow y = \uparrow (x \vee y) \in q\mathcal{B}'$.

(2) Suppose $\emptyset \neq \uparrow x \cap \uparrow y \subseteq \mathord{\uparrow\!\!\uparrow} z \subseteq \uparrow z \in q\mathcal{B}'$. Define $A = \{u \vee v : u \in \mathord{\downarrow\!\!\downarrow} x \text{ and } v \in \mathord{\downarrow\!\!\downarrow} y\}$. By the continuity of P, it is clear that A is directed and $\sup A = x \vee y \in \mathord{\uparrow\!\!\uparrow} z$. Then there are $u \in \mathord{\downarrow\!\!\downarrow} x$ and $v \in \mathord{\downarrow\!\!\downarrow} y$ such that $u \vee v \in \mathord{\uparrow\!\!\uparrow} z$. This shows that there are $\uparrow u, \uparrow v \in q\mathcal{B}'$ such that $\uparrow x \subseteq \mathord{\uparrow\!\!\uparrow} u = \text{int}_{\sigma(P)} \uparrow u$, $\uparrow y \subseteq \mathord{\uparrow\!\!\uparrow} v = \text{int}_{\sigma(P)} \uparrow v$ and $\uparrow x \cap \uparrow y \subseteq \mathord{\uparrow\!\!\uparrow} u \cap \mathord{\uparrow\!\!\uparrow} v \subseteq \uparrow u \cap \uparrow v = \uparrow (u \vee v) \subseteq \mathord{\uparrow\!\!\uparrow} z$.

So, the super-compact quasi-base $q\mathcal{B}'$ has the consistently coherent property. Since $\sigma(P)$ is sober, we have $(Pt(q\mathcal{B}'^\circ), \subseteq) \cong (P, \leq)$, as desired. □

Definition 5.3. A super-compact quasi-base $q\mathcal{B}$ for a topological space X is said to have the *coherent property* if for all $U, V \in q\mathcal{B}$,

(1) $U \cap V \in q\mathcal{B}$;
(2) If $U \cap V \subseteq \text{int } W \subseteq W \in q\mathcal{B}$, then there are $U_1, V_1 \in q\mathcal{B}$ such that $U \subseteq \text{int } U_1$, $V \subseteq \text{int } V_1$ and $U \cap V \subseteq \text{int } U_1 \cap \text{int } V_1 \subseteq U_1 \cap V_1 \subseteq \text{int } W$.

Theorem 5.4. *Let $(X, \mathcal{O}(X))$ be a sober space with a super-compact quasi-base $q\mathcal{B}$ and $X \in q\mathcal{B}$. If $q\mathcal{B}$ has the coherent property, then $(Pt(q\mathcal{B}^\circ), \subseteq)$ is a continuous lattice. Conversely, any continuous lattice can be obtained in this way.*

Proof. It follows from Theorem 5.2 that $(Pt(q\mathcal{B}^\circ), \subseteq)$ is a bc-domain. To show that $(Pt(q\mathcal{B}^\circ), \subseteq)$ is a complete lattice, we need to show that each finite subset of $(Pt(q\mathcal{B}^\circ), \subseteq)$ has a supremum. Let $\alpha, \beta \in Pt(q\mathcal{B}^\circ)$. By Proposition 4.8, $\alpha = \bigcup_{\text{int } U \in \alpha} \Uparrow U$ and $\beta = \bigcup_{\text{int } V \in \beta} \Uparrow V$ are both directed unions. To show that α and β has a supremum in $(Pt(q\mathcal{B}^\circ), \subseteq)$, it suffices to show that for all int $U \in \alpha$ and int $V \in \beta$, formal points $\Uparrow U$ and $\Uparrow V$ has a supremum. Since the super-compact quasi-base $q\mathcal{B}$ has the coherent property, we have $\emptyset \neq U \cap V \in q\mathcal{B}$. By Proposition 4.7, $\Uparrow (U \cap V) \in Pt(q\mathcal{B}^\circ)$. By the proof of Theorem 5.2, $\Uparrow (U \cap V)$ is the supremum of $\Uparrow U$ and $\Uparrow V$ and thus α and β has a supremum. To sum up, $(Pt(q\mathcal{B}^\circ), \subseteq)$ is a continuous lattice.

Conversely, let P be a continuous lattice. It is easy to verify that $q\mathcal{B}' = \{\uparrow x : x \in P\}$ is a super-compact quasi-base for $\sigma(P)$. Clearly, $q\mathcal{B}'$ has the coherent property and contains P. Since P is a continuous lattice, $(Pt(q\mathcal{B}'^\circ), \subseteq) \cong (P, \leq)$, as desired. □

Definition 5.5. Let X be a topological space with a super-compact quasi-base $q\mathcal{B}$. For all $x \in X$, define $q\mathcal{B}(x) = \{U : x \in \text{int } U \subseteq U \in q\mathcal{B}\}$. Then the quasi-base $q\mathcal{B}$ is said to have the *locally coherent property* if for all $U, V \in q\mathcal{B}(x)$,

(1) The infimum of U and V in $q\mathcal{B}(x)$ exists, denoted by $U \cap_x V$;
(2) If $U \cap_x V \subseteq \text{int } W \subseteq W \in q\mathcal{B}(x)$, then there are $U_1, V_1 \in q\mathcal{B}(x)$ such that $U \subseteq \text{int } U_1$, $V \subseteq \text{int } V_1$ and $U \cap_x V \subseteq \text{int } (U_1 \cap_x V_1) \subseteq U_1 \cap_x V_1 \subseteq \text{int } W$.

Theorem 5.6. *Let $(X, \mathcal{O}(X))$ be a sober space with a super-compact quasi-base $q\mathcal{B}$ and $X \in q\mathcal{B}$. If $q\mathcal{B}$ has the locally coherent property, then $(Pt(q\mathcal{B}^\circ), \subseteq)$ is an L-domain with a bottom. Conversely, any L-domain with a bottom can be obtained in this way.*

Proof. Clearly, the set $\{X\}$ is a bottom of $(Pt(q\mathcal{B}^\circ), \subseteq)$. By Theorem 4.6, $(Pt(q\mathcal{B}^\circ), \subseteq)$ is a continuous domain. We need to show that every principal ideal of $(Pt(q\mathcal{B}^\circ), \subseteq)$ is a complete lattice. Let $\gamma \in Pt(q\mathcal{B}^\circ)$. Since X is sober, there is $x \in X$ such that $\phi(x) = \gamma$, where the map ϕ is defined as in Proposition 4.8. To show that the principal ideal $\downarrow \gamma$ of $Pt(q\mathcal{B}^\circ)$ is a complete lattice, it suffices to show that for all $\alpha, \beta \in \downarrow \gamma$, the supremum of α and β in $\downarrow \gamma$ exists. By Proposition 4.8, $\alpha = \bigcup_{\mathrm{int}\, U \in \alpha} \Uparrow U$ and $\beta = \bigcup_{\mathrm{int}\, V \in \beta} \Uparrow V$ are both directed unions. So, to show that α and β has a supremum in $\downarrow \gamma$, it suffices to show that for all $\mathrm{int}\, U \in \alpha$ and $\mathrm{int}\, V \in \beta$, formal points $\Uparrow U$ and $\Uparrow V$ has a supremum in $\downarrow \gamma$. Since γ is an upper bound of α and β, we have $\mathrm{int}\, U$, $\mathrm{int}\, V \in \gamma = \phi(x)$. This implies that $x \in \mathrm{int}\, U \cap \mathrm{int}\, V$ and thus $U, V \in q\mathcal{B}(x)$, where $q\mathcal{B}(x)$ is defined as in Definition 5.5. Since the super-compact quasi-base $q\mathcal{B}$ has the locally coherent property, the infimum $U \cap_x V$ of U and V in $q\mathcal{B}(x)$ exists. By Proposition 4.7, $\Uparrow (U \cap_x V) \in Pt(q\mathcal{B}^\circ)$. Clearly, $\Uparrow (U \cap_x V)$ is an upper bound of $\Uparrow U$ and $\Uparrow V$ in $\downarrow \gamma$. Let $\xi \in Pt(q\mathcal{B}^\circ)$ be any upper bound of $\Uparrow U$ and $\Uparrow V$ in $\downarrow \gamma$. For all $\mathrm{int}\, W \in \Uparrow (U \cap_x V)$, we have $U \cap_x V \subseteq \mathrm{int}\, W \subseteq W \in q\mathcal{B}(x)$. By Definition 5.5, there are $U_1, V_1 \in q\mathcal{B}(x)$ such that $U \subseteq \mathrm{int}\, U_1$, $V \subseteq \mathrm{int}\, V_1$ and $U \cap_x V \subseteq \mathrm{int}\, (U_1 \cap_x V_1) \subseteq U_1 \cap_x V_1 \subseteq \mathrm{int}\, W$. Since $\mathrm{int}\, U_1 \in \Uparrow U$, $\mathrm{int}\, V_1 \in \Uparrow V$ and ξ is a filter of $(q\mathcal{B}^\circ, \subseteq)$, we have $\mathrm{int}\, U_1, \mathrm{int}\, V_1 \in \xi$ and there is some $S \in q\mathcal{B}$ such that $\mathrm{int}\, S \in \xi$ and $\mathrm{int}\, S \subseteq \mathrm{int}\, U_1 \cap \mathrm{int}\, V_1$. By Definition 4.3 (3), there is a family $\{B_i \in q\mathcal{B} : i \in I\}$ such that $\mathrm{int}\, S = \bigcup_{i \in I} \mathrm{int}\, B_i = \bigcup_{i \in I} B_i$. Since ξ is a formal point, there is $i_0 \in I$ such that $\mathrm{int}\, B_{i_0} \in \xi$ and thus $x \in \mathrm{int}\, B_{i_0} \subseteq B_{i_0} \subseteq \mathrm{int}\, S \subseteq \mathrm{int}\, U_1 \cap \mathrm{int}\, V_1 \subseteq U_1 \cap V_1$. This shows that $B_{i_0} \subseteq U_1 \cap_x V_1$ and $\mathrm{int}\, B_{i_0} \subseteq \mathrm{int}\, (U_1 \cap_x V_1) \subseteq U_1 \cap_x V_1 \subseteq \mathrm{int}\, W$. So, $\mathrm{int}\, W \in \xi$ and $\Uparrow (U \cap_x V) \subseteq \xi$, showing that $\Uparrow (U \cap_x V)$ is the supremum of $\Uparrow U$ and $\Uparrow V$ in $\downarrow \gamma$. So, α and β has a supremum in $\downarrow \gamma$ and $\downarrow \gamma$ is a complete lattice. To sum up, $(Pt(q\mathcal{B}^\circ), \subseteq)$ is an L-domain with a bottom.

Conversely, let P be an L-domain with a bottom. It is clear that $q\mathcal{B}' = \{\uparrow x : x \in P\}$ is a super-compact quasi-base for $\sigma(P)$ and $P \in q\mathcal{B}'$. Next we show that $q\mathcal{B}'$ has the locally coherent property. Let $x, y, z \in P$ and let $\uparrow y$, $\uparrow z \in q\mathcal{B}'(x) := \{\uparrow u : x \in \mathord{\Uparrow} u = \mathrm{int}_{\,\sigma(P)} \uparrow u \subseteq \uparrow u \in q\mathcal{B}'\}$.

(1) Clearly, $y \ll x$ and $z \ll x$. Since P is an L-domain, the principal ideal $\downarrow x$ is a complete lattice. So, the supremum of y and z in $\downarrow x$ exists, denoted by $y \vee_x z$. By the continuity of P, it is clear that $y \vee_x z \ll x$ and thus $\uparrow y \cap_x \uparrow z = \uparrow (y \vee_x z) \in q\mathcal{B}'(x)$;

(2) Suppose $\uparrow y \cap_x \uparrow z = \uparrow (y \vee_x z) \subseteq \mathord{\Uparrow} t \subseteq \uparrow t \in q\mathcal{B}'(x)$. Define $A = \{c \vee_x d : c \in \mathord{\Downarrow} y$ and $d \in \mathord{\Downarrow} z\}$. It follows from the continuity of P that A is directed and $\sup A = y \vee_x z \in \mathord{\Uparrow} t$. Then there are $c \in \mathord{\Downarrow} y$ and $d \in \mathord{\Downarrow} z$ such that $c \vee_x d \in \mathord{\Uparrow} t$. This shows that there are $\uparrow c, \uparrow d \in q\mathcal{B}'(x)$ such that $\uparrow y \subseteq \mathord{\Uparrow} c = \mathrm{int}_{\,\sigma(P)} \uparrow c$, $\uparrow z \subseteq \mathord{\Uparrow} d = \mathrm{int}_{\,\sigma(P)} \uparrow d$ and $\uparrow y \cap_x \uparrow z = \uparrow (y \vee_x z) \subseteq \mathord{\Uparrow} (c \vee_x d) = \mathrm{int}_{\,\sigma(P)} (\uparrow c \cap_x \uparrow d) \subseteq \mathord{\Uparrow} t$. This shows that the super-compact quasi-base $q\mathcal{B}'$ has the locally coherent property. Since $\sigma(P)$ is sober, we have $(Pt(q\mathcal{B}'^\circ), \subseteq) \cong (P, \leq)$, as desired. $\square$

Next, we consider more general sL-domains and L-domains.

Theorem 5.7. *Let X be a sober space with a super-compact quasi-base $q\mathcal{B}$. If $q\mathcal{B}$ has the locally coherent property, then $(Pt(q\mathcal{B}^\circ), \subseteq)$ is an sL-domain. Moreover, if for all $x \in X$, $(q\mathcal{B}(x), \subseteq)$ has a top element B_x with $\mathrm{int}\, B_x = B_x$, then $(Pt(q\mathcal{B}^\circ), \subseteq)$ is an L-domain. Conversely, sL-domains and L-domains can be obtained in a corresponding way.*

Proof. Firstly, by the proof of Theorem 5.6, for all $\alpha, \beta, \gamma \in Pt(q\mathcal{B}^\circ)$ with $\alpha, \beta \in \downarrow \gamma$, the supremum of α and β in $\downarrow \gamma$ exists. Then by Theorem 4.6, we conclude that $(Pt(q\mathcal{B}^\circ), \subseteq)$ is an sL-domain. Secondly, when the additional conditions are fulfilled, we show that every principal ideal of $(Pt(q\mathcal{B}^\circ), \subseteq)$ has a bottom. Since X is sober, there is $x \in X$ such that $\phi(x) = \gamma$, where the map ϕ is defined as in Proposition 4.8. Since $(q\mathcal{B}(x), \subseteq)$ has a largest element B_x with $\mathrm{int}\, B_x = B_x$, it is easy to show that $\Uparrow B_x = \{B_x\}$ is a formal point and thus is the bottom of $\downarrow \gamma$. So, in this case, $(Pt(q\mathcal{B}^\circ), \subseteq)$ is an L-domain.

Conversely, let P be an sL-domain. By the proof of Theorem 5.6, $q\mathcal{B}' = \{\uparrow x : x \in P\}$ is a super-compact quasi-base with locally coherent property for $\sigma(P)$. In addition, if P is an L-domain, then for all $x \in P$, since $\downarrow x$ is a complete lattice, there is a least element in $\downarrow x$, denoted by $\perp_x$. Clearly, $\uparrow \perp_x$ is Scott open. It is straightforward to verify that $\uparrow \perp_x$ is the largest element of $q\mathcal{B}'(x) = \{\uparrow u : x \in \mathord{\Uparrow} u = \mathrm{int}_{\,\sigma(P)} \uparrow u \subseteq \uparrow u \in q\mathcal{B}'\}$. Since $\sigma(P)$ is sober, we have $(Pt(q\mathcal{B}'^\circ), \subseteq) \cong (P, \leq)$, as desired. $\square$

6. Quasi-formal points and algebraic domains

To deal with algebraic domains effectively, we introduce the new concept of quasi-formal points as follows.

Definition 6.1. Let X be a topological space with a quasi-base $q\mathcal{B}$. The set $Pt^q(q\mathcal{B})$ of *quasi-formal points* of the topology $\mathcal{O}(X)$ is defined to be all the filters $q\alpha$ of $q\mathcal{B}$ (in the set inclusion order) satisfying for all $V \in q\alpha$ and $V_i \in q\mathcal{B}(i \in I)$ with $V \subseteq \bigcup_{i\in I} \operatorname{int} V_i$, there is $i_0 \in I$ such that $V_{i_0} \in q\alpha$.

Example 6.2. Let $I = [0, 1]$ be the unit interval. It is clear that $q\mathcal{B} = \{[x, 1] : x \in [0, 1]\}$ is a quasi-base for the Scott topology over I. Then the set of formal points $Pt(q\mathcal{B}^\circ) = \{\{(x, 1] : 0 \le x < a\} : a \in (0, 1]\} \cup \{\{[0, 1]\}\}$ and the set of quasi-formal points $Pt^q(q\mathcal{B}) = \{\{[x, 1] : 0 \le x < a\} : a \in (0, 1]\} \cup \{\{[x, 1] : 0 \le x \le a\} : a \in [0, 1]\}$.

Proposition 6.3. *Let X be a sober space with a super-compact quasi-base $q\mathcal{B}$ and α be a formal point w.r.t. the base $q\mathcal{B}^\circ$. Then $q\alpha := \{U \in q\mathcal{B} : \operatorname{int} U \in \alpha\}$ is a quasi-formal point w.r.t. the quasi-base $q\mathcal{B}$. That is to say, every formal point can be viewed as a quasi-formal point in some sense.*

Proof. Since X is sober, there is $x \in X$ such that $\phi(x) = \alpha$, where the map ϕ is defined as in Proposition 4.8.

(1) Clearly, $\emptyset \notin q\alpha$ and $q\alpha \neq \emptyset$.

(2) Let $U, V \in q\alpha$. Then we have $\operatorname{int} U, \operatorname{int} V \in \alpha$. Since $\alpha \in Pt(q\mathcal{B}^\circ)$, there is $W \in q\mathcal{B}$ such that $\operatorname{int} W \in \alpha = \phi(x)$ and $x \in \operatorname{int} W \subseteq \operatorname{int} U \cap \operatorname{int} V$. By Definition 4.3(3), there is $B \in q\mathcal{B}$ such that $x \in \operatorname{int} B \subseteq B \subseteq \operatorname{int} W \subseteq \operatorname{int} U \cap \operatorname{int} V \subseteq U \cap V$. This shows that $\operatorname{int} B \in \alpha$ and $U \cap V \supseteq B \in q\alpha$.

(3) Let $V \in q\alpha$ and $V_i \in q\mathcal{B}(i \in I)$ with $V \subseteq \bigcup_{i\in I} \operatorname{int} V_i$. Since $\operatorname{int} V \in \alpha$ and α is a formal point, there is $i_0 \in I$ such that $\operatorname{int} V_{i_0} \in \alpha$ and $V_{i_0} \in q\alpha$.

To sum up, $q\alpha$ is a quasi-formal point w.r.t. the quasi-base $q\mathcal{B}$, as desired. □

Theorem 6.4. *Let X be a topological space with a super-compact quasi-base $q\mathcal{B}$. Then $(Pt^q(q\mathcal{B}), \subseteq)$ is an algebraic domain. Furthermore, For all $U \in q\mathcal{B}$, let $\Uparrow_q U := \{V \in q\mathcal{B} : U \subseteq V\}$. Then $\{\Uparrow_q U : U \in q\mathcal{B}\}$ is the set of all compact elements of $(Pt^q(q\mathcal{B}), \subseteq)$. Conversely, any algebraic domain can be obtained in this way.*

Proof. Let $\{q\alpha_i : i \in I\}$ be a directed family of quasi-formal points w.r.t the quasi-base $q\mathcal{B}$. Then it is straightforward to show that $\bigcup_{i\in I} q\alpha_i$ is a quasi-formal point and thus $\bigcup_{i\in I} q\alpha_i = \sup_{i\in I} q\alpha_i$ is the supremum of the family. So, $(Pt^q(q\mathcal{B}), \subseteq)$ is a dcpo.

For each $U \in q\mathcal{B}$. It follows from the definition of quasi-formal points that $\Uparrow_q U = \{V \in q\mathcal{B} : U \subseteq V\}$ is a quasi-formal point. For any directed family $\{q\alpha_i : i \in I\}$ of quasi-formal points with $\sup_{i\in I} q\alpha_i = \bigcup_{i\in I} q\alpha_i \supseteq \Uparrow_q U$, there is $i_0 \in I$ such that $U \in q\alpha_{i_0}$ because $U \in \Uparrow_q U$. So, $\Uparrow_q U \subseteq q\alpha_{i_0}$ and $\Uparrow_q U$ is a compact element. It is easy to see that $q\alpha = \bigcup_{U\in q\alpha} \Uparrow_q U$ for any quasi-formal point $q\alpha$. Since for any compact element $q\beta = \bigcup_{V\in q\beta} \Uparrow_q V$ is a directed union, there is $V_0 \in q\beta$ such that $q\beta = \Uparrow_q V_0$. These imply that $K(Pt^q(q\mathcal{B})) = \{\Uparrow_q U : U \in q\mathcal{B}\}$ and $(Pt^q(q\mathcal{B}), \subseteq)$ is an algebraic domain.

Conversely, let P be an algebraic domain. Clearly, $\mathcal{B}' = \{\uparrow x : x \in K(P)\}$ is both a base consisting of super-compact open sets and a super-compact quasi-base for $\sigma(P)$. So, $(Pt^q(\mathcal{B}'), \subseteq) = (Pt(\mathcal{B}'), \subseteq) \cong (P, \le)$, as desired. □

Corollary 6.5 (*See [10]*). *Let $(X, \mathcal{O}(X))$ be a super-coherent topological space with a base $\mathcal{B}$ consisting of super-compact open sets. Then $(Pt(\mathcal{B}), \subseteq)$ is an algebraic domain.*

Proof. By Remark 4.4, the family $\mathcal{B}^* = \{U \in \mathcal{B} : U \neq \emptyset\}$ is both a base consisting of super-compact open sets and a super-compact quasi-base of the space X. In this case, the quasi-formal points w.r.t the quasi-base $\mathcal{B}^*$ are precisely the formal points w.r.t the base $\mathcal{B}^*$. So, by Theorem 6.4, $(Pt(\mathcal{B}^*), \subseteq) = (Pt^q(\mathcal{B}^*), \subseteq)$ is an algebraic domain. Since $\mathcal{B}$ and $\mathcal{B}^*$ are both bases, by Lemma 4.1(1), we have $(Pt(\mathcal{B}), \subseteq) \cong (Pt(\mathcal{B}^*), \subseteq)$ is an algebraic domain, as desired. □

Corollary 6.6. *Let $(X, \mathcal{O}(X))$ be a sober space with a base $\mathcal{B}$ consisting of non-empty super-compact open sets and $X \in \mathcal{B}$. If $\mathcal{B}$ has the consistently coherent property (resp., coherent property, locally coherent property), then $(Pt(\mathcal{B}), \subseteq)$ is a(n) Scott domain (resp., algebraic lattice, algebraic L-domain with a bottom). Conversely, any Scott domain (resp., algebraic lattice, algebraic L-domain with a bottom) can be obtained in a corresponding way.*

Proof. By Corollary 6.5, $(Pt(\mathcal{B}), \subseteq)$ is an algebraic domain. By Remark 4.4 and the proof of Theorem 5.2 (resp., Theorems 5.4 and 5.6), $(Pt(\mathcal{B}), \subseteq) = (Pt^q(\mathcal{B}), \subseteq)$ is a(n) Scott domain (resp., algebraic lattice, algebraic L-domain with a bottom).

Conversely, let P be a(n) Scott domain (resp., algebraic lattice, algebraic L-domain with a bottom). It is straightforward to verify that $\mathcal{B}' = \{\uparrow x : x \in K(P)\}$ is a base consisting of non-empty super-compact open sets for $\sigma(P)$. Clearly, $P \in \mathcal{B}'$ and $\mathcal{B}'$ has the consistently coherent property (resp., coherent property, locally coherent property). Since $\sigma(P)$ is sober, we have $(Pt(\mathcal{B}'), \subseteq) \cong (P, \leq)$, as desired. □

Corollary 6.7. *Let $(X, \mathcal{O}(X))$ be a sober space with a base $\mathcal{B}$ consisting of non-empty super-compact open sets. If $\mathcal{B}$ has the locally coherent property, then $(Pt(\mathcal{B}), \subseteq)$ is an algebraic sL-domain. In addition, if for all $x \in X$, the local base $(\mathcal{B}(x), \subseteq)$ has a largest element B_x, then $(Pt(\mathcal{B}), \subseteq)$ is an algebraic L-domain. Conversely, any algebraic sL-domain and algebraic L-domain can be obtained in a corresponding way.*

Proof. By Corollary 6.5, $(Pt(\mathcal{B}), \subseteq)$ is an algebraic domain. By Remark 4.4 and the proof of Theorem 5.7, $(Pt(\mathcal{B}), \subseteq) = (Pt^q(\mathcal{B}), \subseteq)$ is an algebraic (s)L-domain.

Conversely, let P be an algebraic sL-domain. It is straightforward to verify that $\mathcal{B}' = \{\uparrow x : x \in K(P)\}$ is a base consisting of non-empty super-compact open sets for $\sigma(P)$. Clearly, $\mathcal{B}'$ has the locally coherent property. In addition, if P is an algebraic L-domain, then for all $x \in P$, since $\downarrow x$ is a complete lattice, there is a least element in $\downarrow x$, denoted by $\perp_x$. Clearly, $\perp_x \in K(P)$ and $\uparrow \perp_x$ is Scott open. It is straightforward to verify that $\uparrow \perp_x$ is the largest element of $\mathcal{B}'(x) = \{\uparrow u : x \in \uparrow u \in \mathcal{B}'\}$. Since $\sigma(P)$ is sober, $(Pt(\mathcal{B}'), \subseteq) \cong (P, \leq)$, as desired. □

Remark 6.8. Let P be a continuous domain. Clearly, $q\mathcal{B}' = \{\uparrow x : x \in P\}$ is a super-compact quasi-base for $\sigma(P)$. It is straightforward to show that for each $I \in Idl(P)$, $q\alpha(I) := \{\uparrow x : x \in I\}$ is a quasi-formal point, and that for each quasi-formal point $q\alpha$, $I(q\alpha) := \{x : \uparrow x \in q\alpha\}$ is an ideal of P. With these, we immediately have $(Pt^q(q\mathcal{B}'), \subseteq) \cong (Idl(P), \subseteq)$. By Theorem 4.2, $(P, \leq) \cong (Pt(q\mathcal{B}'^{\circ}), \subseteq)$ and thus $(Idl(P), \subseteq) \cong (Idl(Pt(q\mathcal{B}'^{\circ})), \subseteq)$. So we have $(Pt^q(q\mathcal{B}'), \subseteq) \cong (Idl(Pt(q\mathcal{B}'^{\circ})), \subseteq)$.

Remark 6.9. (1) Given a super-compact quasi-base $q\mathcal{B}$ for a super-coherent space, generally $(Pt^q(q\mathcal{B}), \subseteq) \ncong (Idl(Pt(q\mathcal{B}^{\circ})), \subseteq)$. For example, let P be an algebraic domain. Then $(P, \sigma(P))$ is a super-coherent space. Clearly, $\mathcal{B}' = \{\uparrow x : x \in K(P)\}$ is both a base and a quasi-base consisting of super-compact open sets for $\sigma(P)$. Then $(Pt^q(\mathcal{B}'), \subseteq) = (Pt(\mathcal{B}'), \subseteq) \cong (P, \leq)$. However, generally $(P, \leq) \ncong (Idl(P), \subseteq)$.

(2) Given two super-compact quasi-bases $q\mathcal{B}_1$ and $q\mathcal{B}_2$ for a super-coherent space, by (1) and Remark 6.8, generally $(Pt^q(q\mathcal{B}_1), \subseteq) \ncong (Pt^q(q\mathcal{B}_2), \subseteq)$.

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